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BOUNDARY OPERATOR ALGEBRAS FOR FREE UNIFORM TREE LATTICES

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Abstract. Let X be a finite connected graph, each of whose vertices has degree at least three. The fundamental group Γ of X is a free group and acts on the universal covering tree Δ and on its boundary $\partial\Delta$, endowed with a natural topology and Borel measure. The crossed product C^*-algebra $C(\partial\Delta) \rtimes \Gamma$ depends only on the rank of Γ and is a Cuntz-Krieger algebra whose structure is explicitly determined. The crossed product von Neumann algebra does not possess this rigidity. If X is homogeneous of degree $q+1$ then the von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ is the hyperfinite factor of type III_λ where $\lambda = 1/q^2$ if X is bipartite, and $\lambda = 1/q$ otherwise.

Introduction

Let Δ be a locally finite tree whose automorphism group $\mathrm{Aut}(\Delta)$ is equipped with the compact open topology. Let Γ be a discrete subgroup of $\mathrm{Aut}(\Delta)$ which acts freely on Δ. That is, no element $g \in \Gamma - \{1\}$ stabilizes any vertex or geometric edge of Δ. Assume furthermore that Γ acts cocompactly on Δ, so that the quotient $\Gamma\backslash\Delta$ is a finite graph. Then Γ is a finitely generated free group and is referred to as a free uniform tree lattice.

Conversely, if X is a finite connected graph and Γ is the fundamental group of X, then Γ is a finitely generated free group and acts freely and cocompactly on the universal covering tree Δ.

It is fruitful to think of the tree Δ as a combinatorial analogue of the Poincaré disc and Γ as an analogue of a Fuchsian group. The group Γ is the free group on γ generators, where $\gamma = 1 - \chi(\Gamma\backslash\Delta)$ and $\chi(\Gamma\backslash\Delta)$ is the Euler-Poincaré characteristic of the quotient graph. Let S be a free set of generators for Γ.

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Define a $\{0,1\}$-matrix A of order 2γ, with entries indexed by elements of $S \cup S^{-1}$, by

$$A(x,y) = \begin{cases} 1 & \text{if } y \neq x^{-1}, \\ 0 & \text{if } y = x^{-1}. \end{cases} \tag{0.1}$$

Notice that the matrix A depends only on the rank of the free group Γ.

The boundary $\partial\Delta$ of the tree Δ is the set of equivalence classes of infinite semi-geodesics in Δ, where equivalent semi-geodesics contain a common sub-semi-geodesic. There is a natural compact totally disconnected topology on $\partial\Delta$ [S, I.2.2]. Denote by $C(\partial\Delta)$ the algebra of continuous complex valued functions on $\partial\Delta$. The full crossed product algebra $C(\partial\Delta) \rtimes \Gamma$ is the universal C^*-algebra generated by the commutative C^*-algebra $C(\partial\Delta)$ and the image of a unitary representation π of Γ, satisfying the covariance relation $f(g^{-1}\omega) = \pi(g)\cdot f \cdot \pi(g)^{-1}(\omega)$ for $f \in C(\partial\Delta)$, $g \in \Gamma$ and $\omega \in \partial\Delta$ [Ped, Chapter 7].

Theorem 1. *Let Δ be a locally finite tree whose vertices all have degree at least three. Let Γ be a free uniform lattice in* $\mathrm{Aut}(\Delta)$*. Then the boundary C^*-algebra $\mathcal{A}(\Gamma) = C(\partial\Delta) \rtimes \Gamma$ depends only on the rank of Γ, and Γ is itself determined by $K_0(\mathcal{A}(\Gamma))$. More precisely,*

1. *$\mathcal{A}(\Gamma)$ is isomorphic to the simple Cuntz-Krieger algebra $\mathcal{O}_A$ associated with the matrix A;*
2. *$K_0(\mathcal{A}(\Gamma)) = \mathbb{Z}^\gamma \oplus \mathbb{Z}/(\gamma-1)\mathbb{Z}$ and the class of the identity $[\mathbf{1}]$ is the generator of the summand $\mathbb{Z}/(\gamma-1)\mathbb{Z}$. Moreover $K_1(\mathcal{A}(\Gamma)) = \mathbb{Z}^\gamma$.*

The algebra $\mathcal{A}(\Gamma)$ satisfies the hypotheses of the classification theorem of [K],[Ph]. Therefore the isomorphism class of the algebra $\mathcal{A}(\Gamma)$ is determined by its K-theory together with the class of the identity in K_0. The fact that the class $[\mathbf{1}]$ in K_0 has order equal to $-\chi(\Gamma\backslash\Delta)$ strengthens the result of [Rob, Section 1] and provides an exact analogy with the Fuchsian case [AD].

Theorem 1 will be proved in Lemmas 1.4 and 2.1 below. The key point in the proof is that the Cuntz-Krieger algebra $\mathcal{O}_A$ is defined uniquely, up to isomorphism, by a finite number of generators and relations [CK], and it is possible to identify these explicitly in $\mathcal{A}(\Gamma)$. The original motivation for this result was the paper of J. Spielberg [Spi], which showed that if Γ acts freely and transitively on the tree Δ then $\mathcal{A}(\Gamma)$ is a Cuntz-Krieger algebra. Higher rank analogues were studied in [RS].

There is a natural Borel measure on $\partial\Delta$ and one may also consider the crossed product von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$. This is the von Neumann algebra arising from the classical group measure space construction of Murray and von Neumann [Su]. In contrast to Theorem 1, the structure of this algebra depends on the tree Δ and on the action

of Γ. For simplicity, only the case where Δ is a homogeneous tree is considered.

Theorem 2. *Let Δ be a homogeneous tree of degree $q+1$, where $q \geq 1$, and let Γ be a free uniform lattice in* $\mathrm{Aut}(\Delta)$. *Then* $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ *is the hyperfinite factor of type* III_λ *where*

$$\lambda = \begin{cases} 1/q^2 & \textit{if the graph } \Gamma\backslash\Delta \textit{ is bipartite,} \\ 1/q & \textit{otherwise.} \end{cases}$$

Theorem 2 will be proved in Section 3. The result could equally well have been stated as a classification of the measure theoretic boundary actions up to orbit equivalence [HO]. The analogous result for a Fuchsian group Γ acting on the circle is that $\mathrm{L}^\infty(S^1) \rtimes \Gamma$ is the hyperfinite factor of type III_1 [Spa].

The special case of Theorem 2 where Γ acts freely and transitively on the vertices of Δ was dealt with in [RR]. In that case q is odd, Γ is the free group of rank $\frac{q+1}{2}$, and $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ is the hyperfinite factor of type $\mathrm{III}_{1/q}$. We remark that R. Okayasu [Ok] constructs similar algebras in a different way, but does not explicitly compute the value of λ.

There is a type map τ defined on the vertices of Δ and taking values in $\mathbb{Z}/2\mathbb{Z}$, defined as follows. Fix a vertex $v_0 \in \Delta$ and let $\tau(v) = d(v_0, v) \pmod 2$, where $d(u,v)$ denotes the usual graph distance between vertices of the tree. The type map is independent of v_0, up to addition of 1 (mod 2). It therefore induces a canonical partition of the vertex set of Δ into two classes, so that two vertices are in the same class if and only if the distance between them is even. An automorphism $g \in \mathrm{Aut}(\Delta)$ is said to be *type preserving* if, for every vertex v, $\tau(gv) = \tau(v)$. The graph $\Gamma\backslash\Delta$ is bipartite if and only if the action of Γ is type preserving.

Let $\mathbb{F}$ be a nonarchimedean local field with residue field of order q. The Bruhat-Tits building associated with $\mathrm{PGL}(2,\mathbb{F})$ is a regular tree Δ of degree $q+1$ whose boundary may be identified with the projective line $\mathbb{P}_1(\mathbb{F})$. If Γ is a torsion free lattice in $\mathrm{PGL}(2,\mathbb{F})$ then Γ is necessarily a free group of rank $\gamma \geq 2$, which acts freely and cocompactly on Δ [S, Chapitres I.3.3, II.1.5], and the results apply to the action of Γ on $\mathbb{P}_1(\mathbb{F})$.

Let $\mathcal{O}$ denote the valuation ring of $\mathbb{F}$. Then $K = \mathrm{PGL}(2,\mathcal{O})$ is an open maximal compact subgroup of $\mathrm{PGL}(2,\mathbb{F})$ and the vertex set of Δ may be identified with the homogeneous space $\mathrm{PGL}(2,\mathbb{F})/K$. If the Haar measure μ on $\mathrm{PGL}(2,\mathbb{F})$ is normalized so that $\mu(K) = 1$, then the covolume $\mathrm{covol}(\Gamma)$ is equal to the number of vertices of $X = \Gamma\backslash\Delta$ and $\gamma - 1 = \frac{(q-1)}{2}\,\mathrm{covol}(\Gamma)$, (c.f. [S, Chapitre II.1.5]).

The action of Γ on Δ is type preserving if and only if Γ is a subgroup of $\mathrm{PSL}(2,\mathbb{F})$. Combining Theorem 1 and Theorem 2, in this special case, yields

Corollary 1. *Let Γ be a torsion free lattice in $\mathrm{PGL}(2,\mathbb{F})$. Using the above notation, the boundary algebras are determined as follows.*

(1) *The C^*-algebra $\mathcal{A}(\Gamma) = C(\mathbb{P}_1(\mathbb{F})) \rtimes \Gamma$ is the unique Cuntz-Krieger algebra satisfying*
$$(K_0(\mathcal{A}(\Gamma)), [\mathbf{1}]) = (\mathbb{Z}^\gamma \oplus \mathbb{Z}/(\gamma-1)\mathbb{Z}, (0,0,\dots,0,1)).$$

(2) *The von Neumann algebra $\mathrm{L}^\infty(\mathbb{P}_1(\mathbb{F})) \rtimes \Gamma$ is the hyperfinite factor of type III_λ where*
$$\lambda = \begin{cases} 1/q^2 & \text{if } \Gamma \subset \mathrm{PSL}(2,\mathbb{F}), \\ 1/q & \text{otherwise.} \end{cases}$$

1. The Cuntz-Krieger algebra

Let Δ be a locally finite tree whose vertices all have degree at least three. The results and terminology of [S] will be used extensively. The edges of Δ are directed edges and each geometric edge of Δ corresponds to two directed edges d and $\overline{d}$. Let Δ^0 denote the set of vertices and Δ^1 the set of directed edges of Δ.

Suppose that Γ is a torsion free discrete group acting freely on Δ : that is no element $g \in \Gamma - \{1\}$ stabilizes any vertex or geometric edge of Δ. Then Γ is a free group [S, I.3.3] and there is an orientation on the edges which is invariant under Γ [S, I.3.1]. Choose such an orientation. This orientation consists of a partition $\Delta^1 = \Delta^1_+ \sqcup \overline{\Delta^1_+}$ and a bijective involution $d \mapsto \overline{d} : \Delta^1 \to \Delta^1$ which interchanges the two components of Δ^1. Each directed edge d has an origin $o(d) \in \Delta^0$ and a terminal vertex $t(d) \in \Delta^0$ such that $o(\overline{d}) = t(d)$.

Assume that Γ acts cocompactly on Δ. This means that the quotient $\Gamma\backslash\Delta$ is a finite connected graph with vertex set $V = \Gamma\backslash\Delta^0$ and directed edge set $E = E_+ \sqcup \overline{E_+} = \Gamma\backslash\Delta^1_+ \sqcup \Gamma\backslash\overline{\Delta^1_+}$. The Euler-Poincaré characteristic of the graph is $\chi(\Gamma\backslash\Delta) = n_0 - n_1$ where $n_0 = \#(V)$ and $n_1 = \#(E_+)$, and Γ is the free group on γ generators, where $\gamma = 1 - \chi(\Gamma\backslash\Delta)$.

Choose a tree T of representatives of $\Delta \pmod \Gamma$; that is a lifting of a maximal tree of $\Gamma\backslash\Delta$. The tree T is finite, since Γ acts cocompactly on Δ. Let S be the set of elements $x \in \Gamma - \{1\}$ such that there exists an edge $e \in \Delta^1_+$ with $o(e) \in T$ and $t(e) \in xT$. Then S is a free set of generators for the free group Γ [S, I.3.3, Théorème 4′] and $\gamma = \#S$. It is clear that S^{-1} is the set of elements $x \in \Gamma - \{1\}$ such that there exists an edge $e \in \Delta^1_-$ with $o(e) \in T$ and $t(e) \in xT$. The map $g \mapsto gT$ is a bijection from Γ onto the set of Γ translates of the tree T in Δ, and these translates are pairwise disjoint [S, I.3.3, Proof of Théorème 4′]. Moreover each vertex of Δ lies in precisely one of the sets gT.

The boundary $\partial\Delta$ of the tree Δ is the set of equivalence classes of infinite semi-geodesics in Δ, where equivalent semi-geodesics agree except on finitely many edges. Also $\partial\Delta$ has a natural compact totally disconnected topology [S, I.2.2]. The group Γ acts on $\partial\Delta$ and one can form the crossed product algebra $C(\partial\Delta) \rtimes \Gamma$. This is the universal C^*-algebra generated by the commutative C^*-algebra $C(\partial\Delta)$ and the image of a unitary representation π of Γ, satisfying the covariance relation

$$f(g^{-1}\omega) = \pi(g) \cdot f \cdot \pi(g)^{-1}(\omega) \tag{1.1}$$

for $f \in C(\partial\Delta)$, $g \in \Gamma$ and $\omega \in \partial\Delta$ [Ped]. This covariance relation implies that for each clopen set $E \subset \partial\Delta$ we have

$$\chi_{gE} = \pi(g) \cdot \chi_E \cdot \pi(g)^{-1}. \tag{1.2}$$

In this equation, χ_E is a continuous function and is regarded as an element of the crossed product algebra via the embedding $C(\partial\Delta) \subset C(\partial\Delta) \rtimes \Gamma$. In the present setup the algebra $C(\partial\Delta) \rtimes \Gamma$ is seen *a posteriori* to be simple. Therefore $C(\partial\Delta)\rtimes\Gamma$ coincides with the reduced crossed product algebra [Ped, 7.7.4] and there is no need to distinguish between them notationally.

Fix a vertex $O \in \Delta$ with $O \in T$. Each $\omega \in \partial\Delta$ has a unique representative semi-geodesic $[O, \omega)$ with initial vertex O. A basic open neighbourhood of $\omega \in \partial\Delta$ consists of those $\omega' \in \partial\Delta$ such that $[O, \omega) \cap [O, \omega') \supset [O, v]$ for some fixed $v \in [O, \omega)$. If $g \in \Gamma - \{1\}$, let Π_g denote the set of all $\omega \in \partial\Delta$ such that $[O, \omega)$ meets the tree gT. Note that Π_g is clopen, since T is finite. The characteristic function p_g of the set Π_g is continuous and so lies in $C(\partial\Delta) \subset C(\partial\Delta) \rtimes \Gamma$. The identity element $\mathbf{1}$ of $C(\partial\Delta) \rtimes \Gamma$ is the constant function defined by $\mathbf{1}(\omega) = 1$, $\omega \in \partial\Delta$.

Lemma 1.1. *If $x, y \in S \cup S^{-1}$ with $x \neq y^{-1}$ then*

(a) $\pi(x)p_{x^{-1}}\pi(x^{-1}) = \mathbf{1} - p_x$;
(b) $\pi(x)p_y\pi(x^{-1}) = p_{xy}$.

Proof. (a) By (1.2), the element $\pi(x)p_{x^{-1}}\pi(x^{-1})$ is the characteristic function of the set

$$\begin{aligned} F_x &= \{x\omega \ ; \ \omega \in \partial\Delta, x^{-1}T \cap [O, \omega) \neq \emptyset\} \\ &= \{x\omega \ ; \ \omega \in \partial\Delta, T \cap [xO, x\omega) \neq \emptyset\} \\ &= \{\omega \in \partial\Delta \ ; \ T \cap [xO, \omega) \neq \emptyset\}. \end{aligned}$$

Now there exists a unique edge $e \in \Delta^1$ such that $o(e) \in T$ and $t(e) \in xT$. If $x \in S$ then $e \in \Delta^1_+$ and if $x \in S^{-1}$ then $e \in \overline{\Delta^1_+}$. Therefore

$$\begin{aligned} \partial\Delta - F_x &= \{\omega \in \partial\Delta \ ; \ T \cap [xO, \omega) = \emptyset\} \\ &= \{\omega \in \partial\Delta \ ; \ xT \cap [O, \omega) \neq \emptyset\} \\ &= \Pi_x, \end{aligned}$$

and the characteristic function of this set is p_x. See Figure 1.

The proof of (b) is an easy consequence of (1.2). □

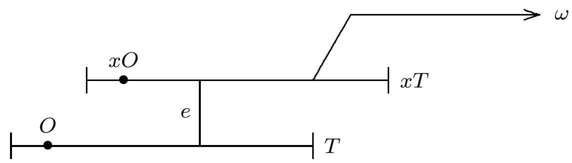

Figure 1. A boundary point $\omega \in \Pi_x$.

Lemma 1.2. *The family of projections $P = \{p_g \ ; \ g \in \Gamma - \{1\}\}$ generates $C(\partial\Delta)$ as a C^*-algebra.*

Proof. We show that P separates points of $\partial\Delta$. Let $\omega_1, \omega_2 \in \partial\Delta$ with $\omega_1 \neq \omega_2$. Let $[O,\omega_1) \cap [O,\omega_2) = [O,v]$, and choose $u \in [v,\omega_1)$ such that $d(v,u)$ is greater than the diameter of T. See Figure 2.

Let $g \in \Gamma$ be the unique element such that $u \in gT$. Then $v \notin gT$ and so $gT \cap [O,\omega_2) = \emptyset$. Therefore $p_g(\omega_1) = 1$ and $p_g(\omega_2) = 0$. □

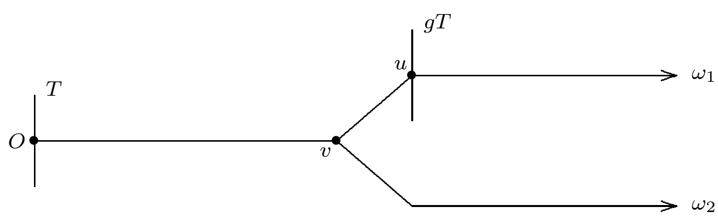

Figure 2. Separation of boundary points.

Lemma 1.3. *The sets of the form Π_x, $x \in S \cup S^{-1}$, are pairwise disjoint and their union is $\partial\Delta$.*

Proof. Given $\omega \in \partial\Delta$, let v be the unique vertex of Δ such that $[O,\omega) \cap T = [O,v]$. Let v' be the vertex of $[O,\omega)$ such that $d(O,v') = d(O,v) + 1$. Then let x be the unique element of $S \cup S^{-1}$ such that $v' \in xT$. See Figure 3. Then $\omega \in \Pi_x$. The sets Π_x, $x \in S \cup S^{-1}$, are pairwise disjoint since the sets xT, $x \in S \cup S^{-1}$, are pairwise disjoint. □

For $x \in S \cup S^{-1}$ define a partial isometry

$$s_x = \pi(x)(\mathbf{1} - p_{x^{-1}}) \in C(\partial\Delta) \rtimes \Gamma.$$

Then, by Lemma 1.1,

$$s_x s_x^* = \pi(x)(\mathbf{1} - p_{x^{-1}})\pi(x^{-1}) = \mathbf{1} - \pi(x)p_{x^{-1}}\pi(x^{-1}) = p_x,$$

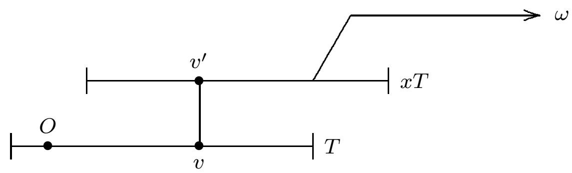

FIGURE 3. Definition of the set Π_x containing ω.

and

$$s_x^* s_x = \mathbf{1} - p_{x^{-1}}.$$

Therefore the elements s_x satisfy the relations

$$s_x^* s_x = \sum_{\substack{y \in S \cup S^{-1} \\ y \neq x^{-1}}} s_y s_y^*. \tag{1.3}$$

Also, it follows from Lemma 1.3 that

$$\mathbf{1} = \sum_{x \in S \cup S^{-1}} p_x = \sum_{x \in S \cup S^{-1}} s_x s_x^*. \tag{1.4}$$

The relations (1.3),(1.4) are precisely the Cuntz-Krieger relations [CK] corresponding to the $\{0,1\}$-matrix A, with entries indexed by elements of $S \cup S^{-1}$, defined by

$$A(x,y) = \begin{cases} 1 & \text{if } y \neq x^{-1}, \\ 0 & \text{if } y = x^{-1}. \end{cases} \tag{1.5}$$

The matrix A depends only on the rank of the free group Γ. Also A is irreducible and not a permutation matrix. It follows that the C^*-subalgebra $\mathcal{A}$ of $C(\partial\Delta) \rtimes \Gamma$ generated by $\{s_x \; ; \; x \in S \cup S^{-1}\}$ is isomorphic to the simple Cuntz-Krieger algebra $\mathcal{O}_A$ [CK]. It remains to show that $\mathcal{A}$ is the whole of $C(\partial\Delta) \rtimes \Gamma$.

Lemma 1.4. *Under the above hypotheses, $C(\partial\Delta) \rtimes \Gamma = \mathcal{A}$.*

Proof. By the discussion above, it is enough to show that

$$\mathcal{A} \supseteq C(\partial\Delta) \rtimes \Gamma.$$

First of all we show that $\mathcal{A} \supseteq \pi(\Gamma)$. It suffices to show that $\pi(x) \in \mathcal{A}$ for each $x \in S \cup S^{-1}$. Now

$$s_{x^{-1}}^* = (\mathbf{1} - p_x)\pi(x) = \pi(x) p_{x^{-1}},$$

by Lemma 1.1. Therefore

$$s_x + s_{x^{-1}}^* = \pi(x)(\mathbf{1} - p_{x^{-1}}) + \pi(x) p_{x^{-1}} = \pi(x). \tag{1.6}$$

It follows that $\pi(x) \in \mathcal{A}$, as required.

Finally, we must show that $\mathcal{A} \supseteq C(\partial\Delta)$. Since $s_x s_x^* = p_x$, it is certainly true that $p_x \in \mathcal{A}$ for all $x \in S \cup S^{-1}$. It follows by induction

from Lemma 1.1(b), that $p_g \in \mathcal{A}$ for all $g \in \Gamma$. Lemma 1.2 now implies that $\mathcal{A} \supseteq C(\partial\Delta)$. $\square$

Example 1.5. Consider the graphs X, Y in Figure 4. Each of them has as universal covering space the 3-homogeneous tree Δ. Each has fundamental group the free group Γ on two generators. Consequently, each gives rise to an action of Γ on Δ. These two actions cannot be conjugate via an element of $\mathrm{Aut}(\Delta)$ because their quotients are not isomorphic as graphs.

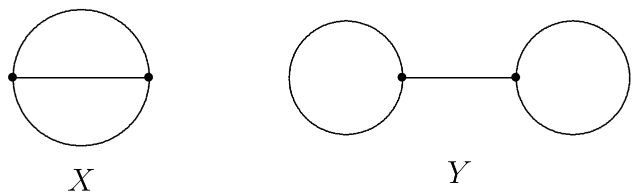

FIGURE 4. The graphs X, Y.

Copies of the free group on two generators, acting with these actions on the corresponding Bruhat-Tits tree Δ, can be found inside $\mathrm{PGL}(2, \mathbb{Q}_2)$, and also inside $\mathrm{PGL}(2, \mathbb{F})$ for any local field $\mathbb{F}$ with residue field of order 2. To see this, note that by [FTN, Appendix, Proposition 5.5] $\mathrm{PGL}(2, \mathbb{Q}_2)$ contains cocompact lattices Γ_1, Γ_2 which act freely and transitively on the vertex set Δ^0 with

$$\Gamma_1 = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b, c \,|\, a^2 = b^2 = c^2 = 1\rangle$$
$$\Gamma_2 = \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z}) = \langle x, d \,|\, d^2 = 1\rangle.$$

The subgroups $\Gamma_X = \langle ab, ac\rangle$ and $\Gamma_Y = \langle x, dxd\rangle$ are both isomorphic to the free group on two generators. Moreover $\Gamma_X\backslash\Delta = X$ and $\Gamma_Y\backslash\Delta = Y$.

2. K-THEORY

Using the results of [C1], it is now easy to determine the K-theory of $\mathcal{A}(\Gamma)$. For each $x \in S \cup S^{-1}$, the element p_x is a projection in $\mathcal{A}(\Gamma)$ and therefore defines an equivalence class $[p_x]$ in $K_0(\mathcal{A}(\Gamma))$. It is shown in [C1] that the classes $[p_x]$ generate $K_0(\mathcal{A}(\Gamma))$. Indeed, let L denote the abelian group with generating set $S \cup S^{-1}$ and relations

$$x = \sum_{\substack{y \in S\cup S^{-1} \\ y \neq x^{-1}}} y \qquad \text{for } x \in S \cup S^{-1}. \tag{2.1}$$

The map $x \mapsto [p_x]$ extends to an isomorphism θ from L onto $K_0(\mathcal{A}(\Gamma))$ [C1]. Moreover $\theta(\varepsilon) = [\mathbf{1}]$, where $\varepsilon = \sum_{x \in S \cup S^{-1}} x$. Now it follows from (2.1) that, for each $x \in S$,

$$\varepsilon = x + x^{-1}. \tag{2.2}$$

Also

$$\varepsilon = \sum_{x \in S}(x + x^{-1}) = \sum_{x \in S} \varepsilon = \gamma\varepsilon.$$

Thus

$$(\gamma - 1)\varepsilon = 0. \tag{2.3}$$

The group L is therefore generated by $S \cup \{\varepsilon\}$, and the relation (2.3) is satisfied.

On the other hand, starting with an abstract abelian group with generating set $S \cup \{\varepsilon\}$ and the relations (2.2), one can make the formal definition $x^{-1} = \varepsilon - x$, for each $x \in S$, and recover the relations (2.1) via

$$\sum_{x \in S}(x + x^{-1}) = \gamma\varepsilon = \varepsilon = x + x^{-1} \qquad \text{for } x \in S.$$

This discussion proves

Lemma 2.1. $K_0(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^\gamma \oplus \mathbb{Z}/(\gamma - 1)\mathbb{Z}$ *via an isomorphism which sends* $[\mathbf{1}]$ *to the generator of* $\mathbb{Z}/(\gamma - 1)\mathbb{Z}$.

It is known that the C^*-algebra $\mathcal{A}(\Gamma)$ is purely infinite, simple, unital and nuclear [CK, C1, C2]. The classification theorem of [K] therefore shows that $\mathcal{A}(\Gamma)$ is determined by its K-theory.

Remark 2.2. If Γ is a torsion free cocompact lattice in $\mathrm{PSL}(2, \mathbb{R})$, so that Γ is the fundamental group of a Riemann surface of genus g, then it is known, [AD, Proposition 2.9], [HN], that $\mathcal{A}(\Gamma) = C(\mathbb{P}_1(\mathbb{R})) \rtimes \Gamma$ is the unique p.i.s.u.n. C^*-algebra whose K-theory is specified by

$$\begin{aligned}
(K_0(\mathcal{A}(\Gamma)), [\mathbf{1}]) &= (\mathbb{Z}^{2g+1} \oplus \mathbb{Z}/(2g-2)\mathbb{Z}, (0, 0, \dots, 0, 1)), \\
K_1(\mathcal{A}(\Gamma)) &= \mathbb{Z}^{2g+1}.
\end{aligned}$$

The proof of this result in [AD] makes use of the Thom Isomorphism Theorem of A. Connes (which has no p-adic analogue) to identify $K_*(\mathcal{A}(\Gamma))$ with the topological K-theory $K^*(\Gamma \backslash \mathrm{PSL}(2, \mathbb{R}))$. It follows from the classification theorem of [K, Ph] that $\mathcal{A}(\Gamma)$ is a Cuntz-Krieger algebra. However there is no apparent dynamical reason for this fact. In contrast, the Cuntz-Krieger algebras of the present article appear naturally and explicitly.

3. The measure theoretic result

The purpose of this section is to prove Theorem 2 of the Introduction. From now on Δ is a homogeneous tree of degree $q+1$, where $q \geq 1$, and Γ is a free uniform lattice in $\mathrm{Aut}(\Delta)$. A similar Theorem could be stated for non-homogeneous trees, and proved by the same methods. The boundary $\partial\Delta$ is endowed with a natural Borel measure. In contrast to the topological result, measure theoretic rigidity for the boundary action fails: the von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ depends on the tree Δ and on the action of Γ. Before proceeding with the proof here are some examples.

Example 3.1. Let Γ be the free group on two generators. Then Γ is the fundamental group of each of the graphs X, Y of Figure 4. The 3-homogeneous tree Δ_3 is the universal covering of both these graphs and there are two corresponding (free, cocompact) actions of Γ on Δ_3. It follows from Theorem 2 that the von Neumann algebra $\mathrm{L}^\infty(\partial\Delta_3) \rtimes \Gamma$ is the hyperfinite factor of type $\mathrm{III}_{1/4}$ in the first case, since X is bipartite, and type $\mathrm{III}_{1/2}$ in the second case, since Y is not bipartite.

The group Γ is also the fundamental group of a bouquet of two circles and the corresponding action of Γ on the 4-homogeneous tree Δ_4 produces the hyperfinite factor of type $\mathrm{III}_{1/3}$. These three actions are the only free and cocompact actions of the free group on two generators on a tree Δ with no vertices of degree ≤ 2.

Remark 3.2. For each $\gamma \geq 2$, it is easy to construct bipartite and non-bipartite 3-homogeneous graphs with fundamental group the free group on γ generators. The corresponding boundary actions are of types $\mathrm{III}_{1/4}$ and $\mathrm{III}_{1/2}$ respectively.

We now proceed with the proof of Theorem 2. As before, fix a vertex $O \in \Delta$. If u, v are vertices in Δ^0, let $[u, v]$ be the directed geodesic path between them, with origin u. The graph distance $d(u, v)$ between u and v is the length of $[u, v]$, where each edge is assigned unit length. If $v \in \Delta^0$ let Ω_v be the clopen set consisting of all $\omega \in \partial\Delta$ such that $v \in [O, \omega)$.

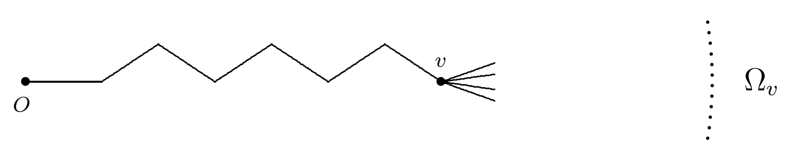

Figure 5. A subset Ω_v of the boundary.

There is a natural Borel measure μ on $\partial\Delta$ defined by $\mu(\Omega_v) = q^{(1-n)}$, where $n = d(O, v)$. The consistency of this definition is easily established, using the fact that there are precisely q vertices w adjacent to

v and not lying on the path $[O, v]$. The set Ω_v is the disjoint union of the corresponding sets Ω_w, each of which has measure q^{-n}. Note that the normalization of this measure is different from that in [FTN]. This is immaterial for the result, but makes the formulae simpler. The measure μ clearly depends on the choice of the vertex O in Δ, but its measure class does not.

Lemma 3.3. *The action of* Γ *on* $\partial\Delta$ *is measure-theoretically free, i.e.*

$$\mu\left(\{\omega \in \partial\Delta : g\omega = \omega\}\right) = 0$$

for all elements $g \in \Gamma - \{e\}$.

Proof. Let $g \in \Gamma - \{e\}$. Since the action of Γ on Δ is free, g is hyperbolic; that is g fixes no point of Δ. It follows that the set $\{\omega \in \partial\Delta : g\omega = \omega\}$ contains exactly two elements and so certainly has measure zero. □

It is well known (and it is an easy consequence of Lemma 3.13 below) that the action of Γ on $\partial\Delta$ is also ergodic. Therefore the von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ is a factor. A convenient reference for this fact and for the classification of von Neumann algebras is [Su]. Most of this section will be devoted to establishing that this factor is of type III_λ, for an appropriate value of λ. This will be done by determining the ratio set of W. Krieger.

Definition 3.4. Let G be a countable group of automorphisms of a measure space (Ω, μ). Define the **ratio set** $r(G)$ to be the subset of $[0, \infty)$ such that if $\lambda \geq 0$ then $\lambda \in r(G)$ if and only if for every $\epsilon > 0$ and measurable set A with $\mu(A) > 0$, there exists $g \in G$ and a measurable set B such that $\mu(B) > 0$, $B \cup gB \subseteq A$ and

$$\left|\frac{d\mu\circ g}{d\mu}(\omega) - \lambda\right| < \epsilon$$

for all $\omega \in B$.

Remark 3.5. The ratio set $r(\Gamma)$ depends only on the quasi-equivalence class of the measure μ. If the action of Γ is ergodic then $r(\Gamma) - \{0\}$ is a subgroup of the multiplicative group of positive real numbers [HO, §I-3, Lemma 14].

In order to compute $r(\Gamma)$, for the action of Γ on $\partial\Delta$, the first step is to find the possible values of the Radon-Nikodym derivatives $\frac{d\mu\circ g}{d\mu}(\omega)$, for $g \in \Gamma$ and $\omega \in \partial\Delta$.

Fix $g \in \Gamma$ and $\omega \in \partial\Delta$. Choose an open set of the form Ω_v with $v \in [O, \omega)$ and $d(O, v) > d(O, gO)$. Such sets Ω_v form a neighbourhood base of ω. Then $v \notin [O, gO]$ (Figure 6), and $g^{-1}\Omega_v = \Omega_{g^{-1}v}$. Since $d(O, g^{-1}v) = d(gO, v)$, we have $\mu(g^{-1}\Omega_v) = q^{-d(gO,v)}$.

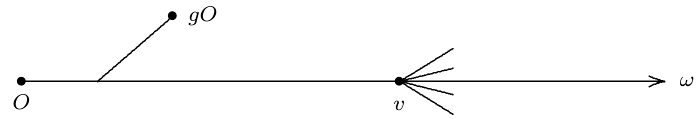

FIGURE 6.

It follows that

$$\frac{d\mu\circ g}{d\mu}(\omega) = \frac{\mu(g^{-1}\Omega_v)}{\mu(\Omega_v)} = \frac{q^{-d(gO,v)}}{q^{-d(O,v)}} = q^{\delta(g,\omega)} \tag{3.1}$$

where $\delta(g,\omega) = d(O,v) - d(gO,v)$. It is clear that $\delta(g,\omega)$ depends only on g and ω, not on the choice of v. In the language of [GH, Chapter 8], $\delta(g,\omega)$ is the Busemann function $\beta_\omega(O,gO)$ relating the horocycles centered at ω containing O, gO respectively. For a fixed vertex v with $d(O,v) > d(O,gO)$, the formula (3.1) remains true for all $\omega \in \Omega_v$.

We have therefore proved

Lemma 3.6. *The values of the Radon-Nikodym derivatives* $\frac{d\mu\circ g}{d\mu}(\omega)$, *for* $g \in \Gamma$ *and* $\omega \in \partial\Delta$, *are given by*

$$\frac{d\mu\circ g}{d\mu}(\omega) = q^{\delta(g,\omega)}$$

Moreover, for each $g \in \Gamma$, *each of these values is attained on a nonempty open subset of* $\partial\Delta$.

These considerations show that

$$r(\Gamma) \subseteq \{q^{\delta(g,\omega)}\,;\, g \in \Gamma, \omega \in \partial\Delta\} \cup \{0\}. \tag{3.2}$$

Since the action of Γ is ergodic, $r(\Gamma) - \{0\}$ is a multiplicative group of positive real numbers [HO, Lemma 14]. What must be done now is to show that the inclusion in (3.2) is in fact an equality. Clearly $r(\Gamma) \neq [0,\infty)$. Therefore if we can show that $r(\Gamma)$ contains a number in the open interval $(0,1)$ then, by [HO, Lemma 15], it must equal $\{\lambda^n\,;\, n \in \mathbb{Z}\} \cup \{0\}$, for some $\lambda \in (0,1)$. By definition, this will show that the action of Γ, and hence the associated von Neumann algebra $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma)$, is of type III_λ.

Before proceeding, it is useful to interpret the situation in terms of the quotient graph $X = \Gamma\backslash\Delta$. In a connected graph X a *proper* path is a path which has no backtracking. That is, no edge $[a,b]$ in the path is immediately followed by its inverse $[b,a]$. A *cycle* is a closed path, which is said to be based at its initial vertex (= final vertex). Note that a proper cycle can have a tail beginning at its base vertex, but that it can have no other tail (Figure 7). Every proper cycle determines a unique tail-less cycle which is obtained by removing the tail. A circuit is a cycle which does not pass more than once through any vertex. There is clearly an upper bound for the possible length of a circuit in X.

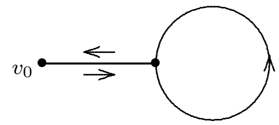

FIGURE 7. A proper cycle with tail, based at v_0.

If $g \in \Gamma$, then the geodesic path $[O, gO]$ in Δ projects to a proper cycle C in the quotient graph $X = \Gamma\backslash\Delta$ based at $v_0 = \Gamma O$. Moreover $d(O, gO)$ is equal to the length $\ell(C)$ of that cycle.

Conversely if C is a proper cycle based at v_0 in the graph X then the homotopy class of C is an element g of the fundamental group Γ of X. The cycle C lifts to a unique proper path in Δ with initial vertex O, namely $[O, gO]$, and $\ell(C) = d(O, gO)$.

In order to prove equality in (3.2) we need the auxiliary concept of the *full group*.

Definition 3.7. Given a group Γ acting on a measure space (Ω, μ), we define the *full group*, $[\Gamma]$, of Γ by

$$[\Gamma] = \{T \in \mathrm{Aut}(\Omega) : T\omega \in \Gamma\omega \text{ for almost every } \omega \in \Omega\}\,.$$

Remark 3.8. The ratio set $r(\Gamma)$ of a countable subgroup Γ of $\mathrm{Aut}(\Omega)$ depends only on the full group in the sense that $r(\Gamma_1) = r(\Gamma_2)$ whenever $[\Gamma_1] = [\Gamma_2]$.

The basis for the proof of equality in (3.2) is the following well known result. It is stated without proof in [HO, I.3].

Lemma 3.9. *Let Γ be a countable group acting ergodically on a measure space Ω. Suppose that the full group $[\Gamma]$ contains an ergodic measure preserving subgroup H.*

If $r \in (0, \infty)$, $g \in \Gamma$ and the set $D = \{\omega \in \Omega\,;\, \frac{d\mu\circ g}{d\mu}(\omega) = r\}$ has positive measure, then $r \in r(\Gamma)$.

Proof. Let A be a measurable subset of Ω with $\mu(A) > 0$. By the ergodicity of H, there exist $h_1, h_2 \in H$ such that the set $B = \{\omega \in A : h_1\omega \in D \text{ and } h_2gh_1\omega \in A\}$ has positive measure.

Let Γ' denote the group generated by h_1, h_2 and Γ. By Remark 3.8, $r(\Gamma') = r(\Gamma)$.

Let $t = h_2gh_1 \in \Gamma'$. By construction, $B \cup tB \subseteq A$. Moreover, since H is measure-preserving,

$$\frac{d\mu\circ t}{d\mu}(\omega) = \frac{d\mu\circ g}{d\mu}(h_1\omega) = r \text{ for all } \omega \in B,$$

since $h_1\omega \in D$. This proves that $r \in r(\Gamma') = r(\Gamma)$, as required. $\square$

In view of Lemma 3.6, all that is now needed in the present setup is the construction of a subgroup H. This will require the following result.

Lemma 3.10. *Let H be subgroup of* $\mathrm{Aut}(\Delta)$. *Suppose that the induced action of H on $\partial\Delta$ is measure preserving and that, for each positive integer n, H acts transitively on the collection of sets*

$$\{\Omega_v : v \in \Delta,\, d(O,v) = n\}.$$

Then H acts ergodically on $\partial\Delta$.

Proof. Suppose that $S_0 \subseteq \partial\Delta$ is a Borel set which is invariant under H and such that $\mu(S_0) > 0$. We show that this implies $\mu(\partial\Delta - S_0) = 0$, thereby establishing the ergodicity of the action.

Define a new measure λ on $\partial\Delta$ by $\lambda(S) = \mu(S \cap S_0)$, for each Borel set $S \subseteq \partial\Delta$. Now, for each $k \in H$,

$$\begin{aligned} \lambda(kS) &= \mu(kS \cap S_0) = \mu(S \cap k^{-1}S_0) \\ &\leq \mu(S \cap S_0) + \mu(S \cap (k^{-1}S_0 - S_0)) \\ &= \mu(S \cap S_0) \\ &= \lambda(S), \end{aligned}$$

and therefore λ is H-invariant.

Fix a positive integer n. The transitivity hypothesis on the action of H implies that

$$\lambda(\Omega_v) = \lambda(\Omega_w)$$

whenever $v, w \in \Delta$, $d(O,v) = d(O,w) = n$. Since $\partial\Delta$ is the union of $q^{(n-1)}(q+1)$ disjoint sets $\{\Omega_v;\, d(O,v) = n\}$, each of which has equal measure with respect to λ, we deduce that, if $d(O,v) = n$,

$$\lambda(\Omega_v) = \frac{\lambda(\partial\Delta)}{q^{(n-1)}(q+1)} = \frac{\mu(S_0)}{q^{(n-1)}(q+1)}.$$

Thus $\lambda(\Omega_v) = c\mu(\Omega_v)$ for every $v \in \Delta$, where $c = \frac{\mu(S_0)}{(q+1)} > 0$. Since the sets Ω_v, $v \in \Delta$, generate the Borel σ-algebra, we deduce that $\lambda(S) = c\mu(S)$ for each Borel set S. Therefore

$$\mu(\partial\Delta - S_0) = c^{-1}\lambda(\partial\Delta - S_0) = c^{-1}\mu((\partial\Delta - S_0) \cap S_0) = 0,$$

thus proving ergodicity. $\square$

It is now convenient to introduce some new terminology.

Definition 3.11. Let X be a finite connected graph. Let v_0 be a vertex of X and let $K \geq 0$. Say that (X, v_0) has property $\mathfrak{L}(K)$ if for any two proper paths P_1, P_2 having the same length n and the same initial vertex v_0, there exists $k \geq 0$, with $k \leq K$, and proper cycles C_1, C_2 based at v_0 such that

(a) The initial segment of C_i is P_i, $i = 1, 2$;
(b) the cycles C_i have the same length $n + k$, $i = 1, 2$.

Property $\mathfrak{L}(K)$ says that any two proper paths of the same length starting at v_0 can be completed to proper cycles of the same length, with a uniform bound on how much must be added to each path.

Lemma 3.12. *Let X be a finite connected graph whose vertices all have degree at least three and let v_0 be a vertex of X. Then (X, v_0) has property $\mathfrak{L}(K)$ for some $K \geq 0$.*

The proof of this technical result is deferred to Section 4. We can now prove that the action of Γ on $\partial\Delta$ satisfies the hypotheses of Lemma 3.9.

Lemma 3.13. *Let Δ be a homogeneous tree of degree $q+1$, where $q \geq 1$ and let Γ be a free uniform lattice in* $\mathrm{Aut}(\Delta)$*. Then, relative to the action of Γ on $\partial\Delta$, the full group $[\Gamma]$ contains an ergodic measure preserving subgroup H.*

Proof. By Lemma 3.10, it suffices to prove the following assertion for any $u, v \in \Delta^0$, with $d(O,u) = d(O,v) = n$.

($\star$) There exists a measure preserving automorphism $\phi \in [\Gamma]$ such that ϕ is almost everywhere a bijection from Ω_u onto Ω_v.

The geodesic paths $[O,u], [O,v]$ in Δ project to proper paths P_u, P_v in X with initial vertex $v_0 = \Gamma O$ and length n. By hypothesis, the graph X has property $\mathfrak{L}(K)$ for some constant $K \geq 0$, relative to v_0. Therefore there exists an integer $k \leq K$ and proper cycles C_u, C_v based at v_0 which have initial segments P_u, P_v respectively and $\ell(P_u) = \ell(P_v) = n + k$.

The cycles C_u, C_v lift to unique geodesic paths $[O, u^*], [O, v^*]$ in Δ with initial segments $[O,u]$, $[O,v]$ respectively and $\Gamma u^* = \Gamma v^* = \Gamma O = v_0$. Since the vertices of X all have degree at least three, we can choose an edge e with $o(e) = v_0$ in X such that e meets the terminal edges of C_u and C_v only at v_0. There are unique vertices $u_1, v_1 \in \Delta^1$ such that $e = \Gamma[u^*, u_1] = \Gamma[v^*, v_1]$. Therefore there exists an element $g \in \Gamma$ such that $g[u^*, u_1] = [v^*, v_1]$. The restriction of the action of g to $\Omega_{u_1} = \{\omega \in \partial\Delta;\ u_1 \in [u^*, \omega)\}$ defines a measure preserving bijection from Ω_{u_1} onto Ω_{v_1}. Define $\phi(\omega) = g(\omega)$ for $\omega \in \Omega_{u_1}$.

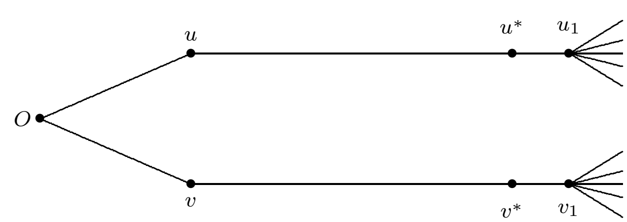

FIGURE 8. Definition of ϕ on Ω_{u_1}.

The set Ω_u is a disjoint union of q^{k+1} sets of the form Ω_w where $d(O,w) = n+k+1$. Each such set therefore has measure $\mu(\Omega_w) = q^{-k-1}\mu(\Omega_u)$. The map ϕ has been defined only on the set Ω_w with $w = u_1$. Therefore ϕ has not yet been defined on a proportion $(1-q^{-k-1})$ of the set Ω_u. Since $k \le K$, the measure of the subset of Ω_u for which ϕ has not yet been defined is at most $(1-q^{-K-1})\mu(\Omega_u)$.

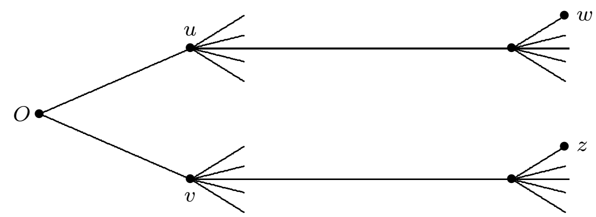

FIGURE 9. Second step in the definition of ϕ.

Now repeat the process on each of the q^{k+1} sets $\Omega_w \subset \Omega_u$, with $d(O,w) = n+k+1$, on which ϕ has not yet been defined. In the preceding argument, replace Ω_u by Ω_w and Ω_v by an appropriate subset Ω_z of Ω_v disjoint from Ω_{v_1}. Note that there is a large amount of arbitrariness in the choice of which Ω_z is to be paired with a particular Ω_w. In each such Ω_w, ϕ is then defined on a subset whose complement in Ω_w has measure at most $(1-q^{-K-1})\mu(\Omega_w)$.

Thus after two steps, ϕ has been defined except on a set of measure at most $(1-q^{-K-1})^2\mu(\Omega_u)$. Continue in this way. After j steps, ϕ has been defined except on a set of measure at most $(1-q^{-K-1})^j\mu(\Omega_u)$.

Since $(1-q^{-K-1})^j \to 0$ as $j \to \infty$, the measure preserving map ϕ is defined almost everywhere on Ω_u, with $\phi(\omega) \in \Gamma\omega$ for almost all $\omega \in \Omega_u$. Finally define ϕ to be the inverse of the map already constructed on Ω_v and the identity map on $\partial\Delta - (\Omega_u \cup \Omega_v)$. The proof of $(\star)$ is complete. $\square$

It follows from Lemmas 3.6, 3.9, and 3.13 that we have equality in (3.2). That is

$$r(\Gamma) = \{q^{\delta(g,\omega)}\,;\, g \in \Gamma, \omega \in \partial\Delta\} \cup \{0\}. \tag{3.3}$$

The final step is to identify this set more precisely. Recall that there is a canonical bipartition of the vertex set of Δ, such that two vertices have the same type if and only if the distance between them is even. The graph $X = \Gamma\backslash\Delta$ is bipartite if and only if the action of Γ is type preserving. Recall also that $\delta(g,\omega) = d(O,v) - d(gO,v)$, for any vertex $v \in [O,\omega) \cap [gO,\omega)$.

Lemma 3.14. *Let Δ be a locally finite tree whose vertices all have degree at least three. Let Γ be a free uniform lattice in* $\mathrm{Aut}(\Delta)$ *and let*

$X = \Gamma\backslash\Delta$. *Then*

$$\{\delta(g,\omega)\,;\, g \in \Gamma, \omega \in \partial\Delta\} = \begin{cases} 2\mathbb{Z} & \textit{if } X \textit{ is bipartite,} \\ \mathbb{Z} & \textit{otherwise.} \end{cases}$$

Proof. Suppose first of all that X is not bipartite. Then X contains a circuit of odd length. Connecting this circuit to v_0 by a minimal path and going around the circuit an appropriate number of times shows that X contains proper cycles based at v_0 of arbitrarily large even and odd lengths.

It follows that we may choose $g \in \Gamma$ such that $d(O, gO) = 2n$, for arbitrarily large n. If $k \in \mathbb{Z}$,with $-n \le k \le n$, let $a \in [O, gO]$ with $d(O, a) = n + k$, $d(gO, a) = n - k$. Choose $\omega \in \partial\Delta$ with $[O, \omega) \cap [gO, \omega) = [a, \omega)$. This is possible since the vertex a has degree at least three (Figure 10). Then $\delta(g, \omega) = n + k - (n - k) = 2k$.

We may also choose $g \in \Gamma$ such that $d(O, gO) = 2n+1$ for arbitrarily large n. If $k \in \mathbb{Z}$,with $-n \le k \le n$, choose $a \in [O, gO]$ with $d(O, a) = n+k$, $d(gO, a) = n+1-k$. Choose $\omega \in \partial\Delta$ with $[O, \omega)\cap[gO, \omega) = [a, \omega)$. Then $\delta(g, \omega) = 2k - 1$.

It follows that the range of the function $\delta(g, \omega)$ is $\mathbb{Z}$.

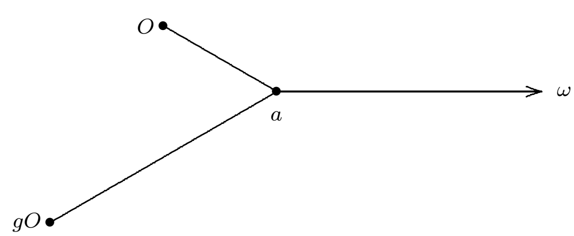

FIGURE 10.

Now suppose that X is bipartite. Then the graph X contains proper cycles of arbitrarily large even length only. The preceding argument shows that the range of the function $\delta(g, \omega)$ is $2\mathbb{Z}$. $\square$

Assume the hypotheses of Theorem 2. Then by (3.3) and Lemma 3.14, the action of Γ on $\partial\Delta$ is of type III_{1/q^2}, if X is bipartite and of type $\mathrm{III}_{1/q}$ otherwise.

In order to complete the proof of Theorem 2, it only remains to prove that the factor $\mathrm{L}^\infty(\partial\Delta) \rtimes \Gamma$ is hyperfinite. By [Z1], this follows from the next result.

Proposition 3.15. *The action of* Γ *on* $\partial\Delta$ *is amenable.*

Proof. The group $G = \mathrm{Aut}(\Delta)$ acts transitively on $\partial\Delta$ [FTN, Chapter I.8]. Fix an element $\omega \in \partial\Delta$, and let $G_\omega = \{g \in G : g\omega = \omega\}$. Then $\partial\Delta \cong G/G_\omega$, and μ corresponds to a measure in the unique quasi-invariant measure class of G/G_ω. The group G_ω is amenable by [FTN, Theorem 8.3]. It follows from [Z2, Corollary 4.3.7] that the action of Γ on G/G_ω is amenable. $\square$

4. Appendix: Proof of a Technical Lemma

This section contains a proof of the technical result, Lemma 3.12. During the course of the proof it will be necessary to concatenate paths. A difficulty arises because two proper paths cannot necessarily be concatenated to produce a proper path. The product path may backtrack at the initial edge of the second path. This problem is overcome by introducing a detour around a proper cycle attached at the initial vertex of the second path. The following auxiliary Lemma will be used to do this.

Lemma 4.1. (Attaching a Loop to an Edge.) *Let X be a finite connected graph whose vertices all have degree at least three. Let e be an edge of X. Then there is a proper cycle L based at the terminal vertex $t(e)$, not passing through e and having length $\ell(L) \leq \delta + \lambda$, where δ is the diameter of X and λ is the maximum length of a circuit in X.*

Proof. The edge e is contained in a maximal tree T in X. Every vertex of X is a vertex of T. Let P be a maximal proper (geodesic) path in T with initial vertex $o(e)$ and initial edge e. Let v be the terminal vertex of P and f the terminal edge. Then v is an endpoint of T. The vertex v has degree at least three. It follows that there are two edges in X other than $\overline{f}$ with initial vertex v. These two edges may both have terminal vertex v (in fact one may be the opposite of the other) or else one or both of them may end at a vertex other than v. However in all cases we may use one or both of these edges together with edges in T to construct a circuit L_0 based at v and not passing through e. The required proper cycle L can the be constructed from $P \cup L_0$. □

Proof of Lemma 3.12. Let δ be the diameter of X and λ the maximum length of a circuit in X. We show that property $\mathfrak{L}(K)$ is satisfied with $K = 10 + 10\delta + 6\lambda$.

Let P_1, P_2 be proper paths in X having the same length n and the same initial vertex v_0. Let p_1, p_2 be the terminal vertices of P_1, P_2 respectively. We must construct proper cycles C_1, C_2 based at v_0 satisfying the conditions of Definition 3.11.

Choose once and for all a path $[p_1, p_2]$ of shortest length between p_1 and p_2. There are two separate cases to consider.

Case 1. The length of $[p_1, p_2]$ is even. Denote this length by $2s$ where $s \geq 0$ and let p_0 be the midpoint of $[p_1, p_2]$. If $s = 0$ then a simpler argument will apply, and produce a smaller bound for the lengths of C_1, C_2, so we assume that $s > 0$.

Choose a path R of minimal length from p_0 to v_0. The cycles C_1, C_2 will be constructed from portions the paths $P_1, P_2, R, [p_1, p_2]$, with loops attached to avoid backtracking. Refer to Figure 11.

Choose an edge e_1 with $o(e_1) = p_1$ such that e_1 is not the initial edge of $[p_1, p_2]$ and $\overline{e}_1$ is not the final edge of P_1. Choose an edge e_2 with

$o(e_2) = p_2$ such that e_2 is not the initial edge of $[p_2, p_1]$ and $\overline{e}_2$ is not the final edge of P_2. For $i = 1, 2$, attach a proper cycle L_i at $t(e_i)$, as in Lemma 4.1.

Assume that the initial edge of R does not meet either of the edges of $[p_1, p_2]$ which contain p_0. Let C_1 be the proper cycle based at v_0 obtained by passing through the following sequence of paths and edges in the order indicated.

$$P_1 \to e_1 \to L_1 \to \overline{e}_1 \to [p_1, p_2] \to e_2 \to L_2 \to \overline{e}_2 \to [p_2, p_0] \to R$$

Similarly, let C_2 be obtained from

$$P_2 \to e_2 \to L_2 \to \overline{e}_2 \to [p_2, p_1] \to e_1 \to L_1 \to \overline{e}_1 \to [p_1, p_0] \to R$$

The proper cycles C_1, C_2 have initial segments P_1, P_2 respectively and have the same length $n+k$, where $k = 4 + \ell(L_1) + \ell(L_2) + 3s + \ell(R) \leq 4 + 2(\delta + \lambda) + \frac{3}{2}\delta + \delta < 4 + 5\delta + 2\lambda$.

Now assume that the initial edge of R meets an edge of $[p_1, p_2]$ which contains p_0. This is precisely the situation illustrated in Figure 11. The cycles C_1, C_2 described above will no longer both be proper, since there will be a backtrack for one of them at the first edge of R. In order to avoid this, choose an edge e_0 with $o(e_0) = p_0$ such that e_0 does not meet either of the edges of $[p_1, p_2]$ containing p_0. Attach a proper cycle L_0 at $t(e_0)$, as in Lemma 4.1. Modify the cycles C_1, C_2 above so that the final part of each becomes

$$\ldots, p_0] \to e_0 \to L_0 \to \overline{e}_0 \to R$$

The proper cycles C_1, C_2 now have the same length $n+k$, where $k < 6 + 6\delta + 3\lambda$.

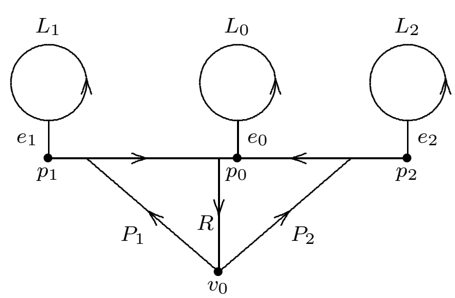

FIGURE 11. Constructing proper cycles of the same length.

CASE 2. The length of $[p_1, p_2]$ is odd. Denote this length by $2s+1$ where $s \geq 0$ and let p_0 be the vertex of $[p_1, p_2]$ with $d(p_0, p_1) = s$, $d(p_0, p_2) = s+1$. The argument that follows will be slightly different, but simpler, if $s = 0$. We therefore again assume that $s > 0$.

Exactly the same argument as in Case 1 shows that there are proper cycles C_1, C_2 based at v_0 and with initial segments P_1, P_2 respectively. The only difference is that $\ell(C_1) = n + k + 1$, $\ell(C_2) = n + k$, where $k < 6 + 6\delta + 3\lambda$.

The cycles will be modified to have the same length by adding to the end of each an appropriate proper cycle based at v_0. The (possibly improper) cycle

$$P_1 \to [p_1, p_2] \to \overline{P}_2$$

has odd length. Deleting appropriate parts of this cycle shows that X contains a circuit C_0 of odd length $2t+1$. (In other words, the graph X is not bipartite.)

Choose a path S_1 of minimal length from v_0 to C_0. Let v_1 be the terminal vertex of S_1. The circuit C_0 is the union of two proper paths C_0^+, C_0^- with lengths $t+1$, t respectively and initial vertex v_1. Let v_2 be the terminal vertex of the paths C_0^+, C_0^-. Choose a path S_2 of minimal length from v_2 to v_1. Add to the end of each of the cycles C_1, C_2 a cycle based at v_0, as indicated below

$$C_1 \to S_1 \to C_0^- \to S_2$$

$$C_2 \to S_1 \to C_0^+ \to S_2$$

The resulting cycles have the same length, namely $n+k+1+t+\ell(S_1)+\ell(S_2) = n+k'$, where $k' \le k+\lambda+2\delta < 6+8\delta+4\lambda$. Either or both of these cycles may have backtracking at v_0 or at v_2 (but not at v_1). If this happens add an edge (and its reverse) and adjoin a loop to both cycles at the relevant vertex as in Lemma 4.1. The resulting cycles are proper (i.e. have no backtracking) and have the same length $n+k''$, where $k'' \le 10+10\delta+6\lambda$. □

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